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$\mathcal{N}=1/2$ super Yang-Mills theory on euclidean $AdS_2 \times S^2$

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Abstract: We study D-branes in the background of euclidean $AdS_2 \times S^2$ with a graviphoton field turned on. As the background is not Ricci flat, the graviphoton field must have both self-dual and antiself-dual parts. This, in general, will break all the supersymmetries on the brane. However, we show that there exists a limit for which one can restore half of the supersymmetries. Further, we show that in this limit, the $\mathcal{N}=1/2$ SYM lagrangian on flat space can be lifted on to the euclidean $AdS_2 \times S^2$ preserving the same amount of supersymmetries as in the flat case. We observe that without the C-dependent terms present in the action this lift is not possible.

Keywords: D-branes, Supersymmetric Effective Theories.

Contents

1. Introduction

Recent studies on string theory in the background of a graviphoton field have revealed new structures on the worldvolume of the corresponding D-branes. In fact, it is found that in such a background the odd coordinates of the superspace turn out to be nonanticommuting. There are, though, two different approaches to the problem. Either, one could insist on preserving the whole $\mathcal{N} = 1$ supersymmetry, as in the work of Ooguri-Vafa [1]. Or, as in [2, 3], one could assume that the anticommutation relations between the odd coordinates on superspace survive the field theory limit. In the latter case, however, one loses half of the supersymmetries, and so it is called $\mathcal{N} = 1/2$ super Yang-Mills theory. Superspace deformations of this kind have been studied in some earlier works [4, 5, 6, 7, 8]. But the fact that it arises in a natural way from string theory was the consequence of recent works [9, 1, 2]. Different aspects of $\mathcal{N} = 1/2$ supersymmetric models have further been studied in [10]–[17]. While the instanton solutions and some nonperturbative effects [18]–[22], along with the generalizations to $\mathcal{N} = 2$, and other interesting features of noncommutative superspace have been explored in [23]–[35].

So far, the study of D-branes in the presence of a graviphoton field has been restricted to flat space times. Roughly speaking, the supergravity field equations imply that for having a flat background, one has to choose a graviphoton field which has a zero energy momentum tensor. In euclidean signature, on the other hand, one way of getting a zero energy momentum tensor is to choose the graviphoton field to be (anti)self-dual. For a typical graviphoton field, however, the energy momentum tensor is not necessarily vanishing. And hence, the supergravity equations would imply that the spacetime cannot be flat. In lorentzian signature, a well known example of this type is $AdS_2 \times S^2$ together with a "self-dual" graviphoton field. In this article, we will be studying the euclidean version of this solution.

Upon rotation to euclidean space, the graviphoton field will have both self-dual and antiself-dual parts. From the standard arguments in string theory it then follows that the anticommutation relations for both right-handed and left-handed odd coordinates on the

D-brane get deformed. Furthermore, this also affects the supersymmetry algebra breaking all the supercharges on the brane. In this article, we take a scaling limit where the self-dual part becomes very large, and at the same time the antiself-dual part goes to zero. The limiting process, however, leaves the energy momentum tensor unaffected and so the $AdS_2 \times S^2$ background is left unchanged. As a result of this limiting half of the supersymmetries can be restored, just as in the flat case. Having the $\mathcal{N} = 1/2$ supersymmetry algebra, we go on to define the corresponding lagrangian on the $AdS_2 \times S^2$ background. In doing so, we will make two interesting observations. Firstly, we observe that for having a supersymmetric lagrangian in this background one actually does need the extra C-dependent terms present in the Seiberg action. In other words, it is not possible to have a pure $\mathcal{N} = 1$ SYM action on $AdS_2 \times S^2$. Secondly, even for having an $\mathcal{N} = 1/2$ supersymmetric lagrangian in this background we need to modify the supersymmetry transformations. However, we show that the modified supersymmetry transformations continue to be a realization of $\mathcal{N} = 1/2$ supersymmetry algebra.

At the end, we comment on the plane wave limit of $AdS_2 \times S^2$ in the lorentzian signature. In this limit the deformation parameter C will have a zero determinant, which allows rotation to a frame where half of the odd coordinates become anticommuting. However, making the corresponding supercharges anticommute is not straightforward. In fact, we find a linear combination of supercharges that anticommute, but then they will not act as derivations.

2. String theory on $AdS_2 \times S^2$ with RR graviphoton background

Consider type-II string theory compactified on a 6-dimensional Calabi-Yau M. Besides the NS-NS fields, we are also considering form fields which come from the compactification of RR form fields on the Calabi-Yau cycles. Among these fields there is a one form field, the graviphoton field, which can be obtained from wrapping RR-forms on cycles of M. In pure spinor notation the field strength of this graviphoton can be shown as follows:

$$P^{\alpha\hat{\beta}} = \frac{1}{2}P^{\mu\nu}(\sigma_{\mu\nu})^{\alpha\hat{\beta}} \tag{2.1}$$

$$P^{\dot{\alpha}\hat{\dot{\beta}}} = \frac{1}{2}P^{\mu\nu}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\hat{\dot{\beta}}}\,, \tag{2.2}$$

where by construction $P^{\alpha\hat{\beta}}$ and $P^{\dot{\alpha}\hat{\dot{\beta}}}$ are self-dual and antiself-dual parts of the graviphoton field, respectively. Here $(\alpha, \dot{\alpha}, \hat{\alpha}, \hat{\dot{\alpha}})$ show the left-handed and right-handed indices of $\mathcal{N} = 2$ fermions.

In euclidean signature, the energy momentum tensor of a self-dual (antiself-dual) graviphoton field is zero, and thus there is no back reaction on the geometry. However, if we consider a typical graviphoton field with both self-dual and antiself-dual parts, there will be a back reaction. In this article, we will be interested in the specific example of euclidean $AdS_2 \times S^2$. We are going to examine the branes in this background, and in the next section write down their effective SYM lagrangian. In lorentzian signature, this background together with a "self-dual" graviphoton field is a well-known supergravity solution, on which the string theory is already studied [36, 37].

Let us start with the string action on flat space, and in the hybrid or pure spinor formalism [37]:

$$S_{\text{flat}} = \frac{1}{\alpha'} \int d^2z \left(\frac{1}{2} \partial_{\bar{z}} x^\mu \partial x_\mu + p_\alpha \partial_{\bar{z}} \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial_{\bar{z}} \bar{\theta}^{\dot{\alpha}} + p_{\hat{\alpha}} \partial \theta^{\hat{\alpha}} + \bar{p}_{\hat{\dot{\alpha}}} \partial \bar{\theta}^{\hat{\dot{\alpha}}} \right) \tag{2.3}$$

where p's are the momentum conjugate to θ's. Following [37], we introduce a new set of variables (and similar definitions for $\hat{d}, \hat{\bar{d}}, \hat{q}$ and $\hat{\bar{q}}$):

$$\begin{aligned} y^\mu &= x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} + i\theta^{\hat{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\hat{\dot{\alpha}}} \\ \bar{q}_{\dot{\alpha}} &= \bar{p}_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_z x_\mu - \theta\theta \partial_z \bar{\theta}_{\dot{\alpha}} + \frac{1}{2} \bar{\theta}_{\dot{\alpha}} \partial_z (\theta\theta) \\ d_\alpha &= -p_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_z x_\mu + \frac{1}{2} \overline{\theta\theta} \partial_z \theta_\alpha - \frac{3}{2} \partial_z (\theta_\alpha \overline{\theta\theta}) \\ q_\alpha &= -p_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_z x_\mu - \overline{\theta\theta} \partial_z \theta_\alpha + \frac{1}{2} \theta_\alpha \partial_z (\overline{\theta\theta}) \\ \bar{d}_{\dot{\alpha}} &= \bar{p}_{\dot{\alpha}} + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_z x_\mu + \frac{1}{2} \theta\theta \partial_z \bar{\theta}_{\dot{\alpha}} - \frac{3}{2} \partial_z (\bar{\theta}_{\dot{\alpha}} \theta\theta) \,. \end{aligned} \tag{2.4}$$

By adding the vertex operator of the graviphoton field, we find the following action:

$$\begin{aligned} S_{\text{flat}} = \frac{1}{\alpha'} \int d^2z \bigg(& \frac{1}{2} \partial_{\bar{z}} y^\mu \partial y_\mu - d_\alpha \partial_{\bar{z}} \theta^\alpha + \bar{q}_{\dot{\alpha}} \partial_{\bar{z}} \bar{\theta}^{\dot{\alpha}} - d_{\hat{\alpha}} \partial \theta^{\hat{\alpha}} + \bar{q}_{\hat{\dot{\alpha}}} \partial \bar{\theta}^{\hat{\dot{\alpha}}} + \\ & + \alpha' P^{\alpha\hat{\beta}} d_\alpha d_{\hat{\beta}} + \alpha' \bar{P}^{\dot{\alpha}\hat{\dot{\beta}}} \bar{d}_{\dot{\alpha}} \bar{d}_{\hat{\dot{\beta}}} \bigg) . \end{aligned} \tag{2.5}$$

It is now straightforward to covariantize the above action, and define it on $AdS_2 \times S^2$ background, as follows

$$\begin{aligned} S = \frac{1}{\alpha'} \int \bigg[& \frac{1}{2} \Pi^c_z \Pi_{\bar{z}c} + d_\alpha \Pi^\alpha_{\bar{z}} + \bar{d}_{\dot{\alpha}} \Pi^{\dot{\alpha}}_{\bar{z}} + d_{\hat{\alpha}} \Pi^{\hat{\alpha}}_{\bar{z}} + \bar{d}_{\hat{\dot{\alpha}}} \Pi^{\hat{\dot{\alpha}}}_{\bar{z}} + \\ & + \alpha' d_\alpha P^{\alpha\hat{\beta}} d_{\hat{\beta}} + \alpha' \bar{d}_{\dot{\alpha}} \bar{P}^{\dot{\alpha}\hat{\dot{\beta}}} \bar{d}_{\hat{\dot{\beta}}} \bigg] , \end{aligned} \tag{2.6}$$

where we have defined $\Pi^A_j \equiv E^A_M \partial_j Z^M$, with E^A_M representing the supervierbein and $Z^M = (x^\mu, \theta^\gamma, \bar{\theta}^{\dot{\gamma}}, \theta^{\hat{\gamma}}, \bar{\theta}^{\hat{\dot{\gamma}}})$. The index A indicates the tangent superspace indices $(c, \alpha, \dot{\alpha}, \hat{\alpha}, \hat{\dot{\alpha}})$. The lowest component of E^c_μ is the vierbein, and the lowest components of E^α_M and $E^{\hat{\alpha}}_M$ are the gravitini which are set to zero in our background (no gravitino). The equations of motion for d_α and $d_{\hat{\alpha}}$ fields read,

$$\begin{aligned} \alpha' P^{\alpha\hat{\beta}} d_{\hat{\beta}} + \Pi^\alpha_{\bar{z}} &= 0 \,, \\ \alpha' P^{\alpha\hat{\beta}} d_\beta - \Pi^{\hat{\alpha}}_z &= 0 \,. \end{aligned} \tag{2.7}$$

Using these equations of motions, we can integrate out the corresponding fields (as well as $\bar{d}_{\dot{\alpha}}$ and $\bar{d}_{\hat{\dot{\alpha}}}$) to find the following action:

$$S = \frac{1}{\alpha'} \int dz d\bar{z} \left[\frac{1}{2} \Pi^c_z \Pi_{\bar{z}c} + \frac{1}{\alpha'} P_{\hat{\alpha}\beta} \Pi^{\hat{\alpha}}_{\bar{z}} \Pi^\beta_z + \frac{1}{\alpha'} P_{\hat{\dot{\alpha}}\dot{\beta}} \Pi^{\hat{\dot{\alpha}}}_{\bar{z}} \Pi^{\dot{\beta}}_z \right] , \tag{2.8}$$

where $P_{\hat{\alpha}\beta}$ and $\bar{P}_{\hat{\dot{\alpha}}\dot{\beta}}$ are the inverses of $P^{\hat{\alpha}\beta}$ and $\bar{P}^{\hat{\dot{\alpha}}\dot{\beta}}$, respectively.

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In the present set up, D-branes can be introduced by a set of consistent open string boundary conditions as follows

$$\theta_\alpha = \theta_{\hat{\alpha}}\,, \qquad d_\alpha = d_{\hat{\alpha}}\,. \tag{2.9}$$

Putting the second equation above in (2.7) implies that

$$\Pi^\alpha_{\bar{z}} = -\Pi^{\hat{\alpha}}_z\,. \tag{2.10}$$

To see the consequence of the above equation, let us first introduce the explicit forms of Π^A_j's. Since the background is fixed, in the expression for the supervierbeins we set the gravitino field to zero, i.e., $\psi^\alpha_\mu = \psi^{\hat{\alpha}}_\mu = 0$, which therefore results to

$$\begin{aligned}
\Pi^{\hat{\alpha}}_z &= \partial_z\theta^{\hat{\alpha}}\,,\\
\Pi^{\alpha}_{\bar{z}} &= \partial_{\bar{z}}\theta^{\alpha}\,,\\
\Pi^c_z &= e^c_\mu\partial_z x^\mu + i\partial_z\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_z\bar{\theta}^{\dot{\alpha}} + i\partial_z\theta^{\hat{\alpha}}\sigma^\mu_{\hat{\alpha}\dot{\hat{\alpha}}}\bar{\theta}^{\dot{\hat{\alpha}}} + i\theta^{\hat{\alpha}}\sigma^\mu_{\hat{\alpha}\dot{\hat{\alpha}}}\partial_z\bar{\theta}^{\dot{\hat{\alpha}}}\\
&= e^c_\mu\partial_z y^\mu\,,\\
\Pi^c_{\bar{z}} &= e^c_\mu\partial_{\bar{z}} x^\mu + i\partial_{\bar{z}}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_{\bar{z}}\bar{\theta}^{\dot{\alpha}} + i\partial_{\bar{z}}\theta^{\hat{\alpha}}\sigma^\mu_{\hat{\alpha}\dot{\hat{\alpha}}}\bar{\theta}^{\dot{\hat{\alpha}}} + i\theta^{\hat{\alpha}}\sigma^\mu_{\hat{\alpha}\dot{\hat{\alpha}}}\partial_{\bar{z}}\bar{\theta}^{\dot{\hat{\alpha}}}\\
&= e^c_\mu\partial_{\bar{z}} y^\mu\,.
\end{aligned} \tag{2.11}$$

Further, equation (2.10) now implies the boundary conditions on the odd coordinates

$$\partial_z\theta^{\hat{\alpha}} = -\partial_{\bar{z}}\theta^\alpha\,, \tag{2.12}$$

which, together with the bosonic boundary conditions, defines a D-brane filling the whole 4-dimensional space.

At this stage, by plugging (2.11) into (2.8), we can write the action in terms of $(y^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}, \theta^{\hat{\alpha}}, \bar{\theta}^{\dot{\hat{\alpha}}})$. In these coordinates, it is easy to calculate the two point function of θ's:

$$\langle\theta^\alpha(z,\bar{z})\theta^\beta(w,\bar{w})\rangle = \frac{\alpha'^2 P^{\alpha\beta}}{2\pi\imath}\log\frac{\bar{z}-w}{z-\bar{w}}\,. \tag{2.13}$$

Following the standard argument about the non-commutativity on the brane, we find a set of non(anti)commutative coordinates on the brane as follows:

$$\{\theta^\alpha, \theta^\beta\} = \alpha'^2 P^{\alpha\beta} \tag{2.14}$$

$$\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \alpha'^2 P^{\dot{\alpha}\dot{\beta}}\,. \tag{2.15}$$

In the field theory limit, where the low energy dynamics of D-branes is studied, we have to consider the limit of $\alpha' \to 0$. However, the nonanticommuting characteristics of D-branes observed above can survive the limit if at the same time we take the limit $P^{\alpha\beta} \to \infty$, holding $C^{\alpha\beta} \equiv \alpha'^2 P^{\alpha\beta}$ fixed. Therefore in such a limit, we expect that the decoupled theory on the worldvolume of the brane to be a deformed SYM theory on a noncommutative curved superspace. However, a simple analogy with the supersymmetry algebra in the flat case shows that the above deformation will break all the supersymmetries.

Although the superspace deformation breaks all the supersymmetries, in the following, we show that there exists a limit where we can restore half of the supersymmetries. To see this, first note that in euclidean signature the energy momentum tensor of the graviphoton field reads

$$T_{\mu\nu} = \frac{1}{2} C^{+}_{\mu\lambda} C^{-}_{\rho\nu}\, g^{\lambda\rho} \,, \tag{2.16}$$

where $C^+(C^-)$ indicate the self-dual (antiself-dual) part of the graviphoton field. Therefore, if the metric is not Ricci flat, the supergravity equations imply that the graviphoton field must have both self-dual and antiself-dual parts. Let us, though, introduce a real parameter k and write $T_{\mu\nu}$ differently

$$T_{\mu\nu} = \frac{1}{2}\left(k C^{+}_{\mu\lambda}\right)\left(\frac{1}{k} C^{-}_{\rho\nu}\right) g^{\lambda\rho} \,. \tag{2.17}$$

Writing $T_{\mu\nu}$ in this way, suggests that we can define a new graviphoton field

$$\tilde{C}_{\mu\nu} = \tilde{C}^{+}_{\mu\nu} + \tilde{C}^{-}_{\mu\nu} = k\, C^{+}_{\mu\nu} + \frac{1}{k}\, C^{-}_{\mu\nu} \,, \tag{2.18}$$

with the same energy momentum tensor as before (i.e., when $k = 1$). Furthermore, this allows us to take a limit where $k \to \infty$, without disturbing the equations of motion or the $AdS_2 \times S^2$ background. The point, however, is that in this limit the antiself-dual part of the graviphoton field goes to zero.[1] Hence, recalling the definitions

$$\begin{aligned} C^{\alpha\beta} &= \frac{1}{2}(\sigma_{\mu\nu})^{\alpha\beta} C^{\mu\nu+} \\ C^{\dot\alpha\dot\beta} &= \frac{1}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta} C^{\mu\nu-} \,, \end{aligned}$$

Eq. (2.15) implies that the usual anticommutation relation between $\bar\theta$ coordinates is restored in this limit, i.e. we will have

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \tag{2.19}$$

$$\{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = 0 \,, \tag{2.20}$$

or in components we will have $C^{\alpha\beta} \sim k/R$, and $C^{\dot\alpha\dot\beta}$ scales to zero for large k. In the next section, we will further show that in the large k limit, the $\mathcal{N} = 1/2$ SYM lagrangian on flat euclidean space [2] can be lifted on to the curved background $AdS_2 \times S^2$.

In the last part of this section, we examine the corresponding euclidean solution of a given solution of supergravity equations of motion in lorentzian signature. Specifically, it is well known that a "self-dual" graviphoton field

$$C = C_{\mu\nu}\, dx^\mu \wedge dx^\nu = 2R\left(\cosh\rho\, d\tau \wedge d\rho + \cos\theta\, d\psi \wedge d\theta\right) , \tag{2.21}$$

together with a metric of $AdS_2 \times S^2$

$$ds^2 = R^2(-\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \cos^2\theta\, d\psi^2 + d\theta^2) \tag{2.22}$$

[1] For the clarity of notation, in the following, we drop the tilde sign of the C field and keep in mind that $C^+(C^-)$ carry a factor of $k\,(1/k)$.

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solve the supergravity equations. Obviously if we rotate to a euclidean signature by sending $\tau \to i\tau$, the rotated metric and graviphoton field continue to solve the supergravity equations in the euclidean signature.[2] Upon Wick rotation, the graviphoton field will have both self-dual and antiself-dual parts

$$C = C^{+} + C^{-}\,, \tag{2.24}$$

where

$$\begin{aligned} C^{+} &= (1+i)Rk\,(\cosh\rho\, d\tau\wedge d\rho + \cos\theta\, d\psi\wedge d\theta) \\ C^{-} &= -(1-i)\frac{R}{2k}\,(\cosh\rho\, d\tau\wedge d\rho - \cos\theta\, d\psi\wedge d\theta)\,, \end{aligned} \tag{2.25}$$

and we have included the k factor for the limiting purposes. It is easy to explicitly check that the above field configuration, together with

$$ds^2 = R^2(\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \cos^2\theta\, d\psi^2 d\theta^2)\,, \tag{2.26}$$

is a solution to the supergravity equations in euclidean signature.[3] Now that we have a consistent euclidean background solution, we can go on to discuss the construction of supersymmetric action in this background.

3. $\mathcal{N} = 1/2$ SYM action on euclidean $AdS_2 \times S^2$

We have chosen to study D-branes on euclidean $AdS_2 \times S^2$ for the following reasons. Firstly, this background, with the corresponding graviphoton field, is a maximally supersymmetric solution admitting maximal number of Killing spinors. Secondly, in the euclidean version, the self-dual and antiself-dual parts of the graviphoton field are independent of each other, and thus can be scaled differently. For constructing the action, we choose the minimal coupling to the background metric. So basically we define the $\mathcal{N} = 1/2$ SYM action of Seiberg [2] on euclidean $AdS_2 \times S^2$ simply by covariantizing the ordinary derivatives both in the action and in the supersymmetry transformations. This will then reduce to the action on flat space in the large R limit. The invariance of the action under the Q supersymmetry, however, is not obvious and we are going to check it in detail. Along the way, we make two important observations. Firstly, we observe that this lift is not possible for pure $\mathcal{N} = 1$ SYM action, and in fact one does need the extra C terms in the action for having invariance under Q. Secondly, even in the case of C-deformed theory, we need to deform the supersymmetry variation of $\overline{\lambda}$ to have a supersymmetric action.

[2] The graviphoton field becomes complex-valued upon Wick rotation. Nevertheless, we could have instead worked with the following real-valued field

$$\begin{aligned} C^{+} &= 2Rk\,(\cosh\rho\, d\tau\wedge d\rho + \cos\theta\, d\psi\wedge d\theta) \\ C^{-} &= -\frac{R}{k}\,(\cosh\rho\, d\tau\wedge d\rho - \cos\theta\, d\psi\wedge d\theta)\,, \end{aligned} \tag{2.23}$$

which also solves the supergravity field equations.

[3] We note that the field C is covariantly constant, and thus satisfies both the Maxwell equations and the Bianchi identity.

To begin with, let us consider the deformed action of Seiberg [2] on a curved D-brane which has filled the euclidean $AdS_2 \times S^2$ space:

$$S_{(C\neq 0)} = \int \sqrt{g}\, d^4x \,\mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\lambda\sigma^{\mu}D_{\mu}\overline{\lambda} - iC^{\mu\nu}F_{\mu\nu}\overline{\lambda\lambda} + \frac{1}{4}|C|^2(\overline{\lambda\lambda})^2\right], \tag{3.1}$$

where $D_\mu \equiv \nabla_\mu + [A_\mu,\ ]$, and ∇_μ is the covariant derivative on the curved background $AdS_2 \times S^2$. We now show that the above action is invariant under the following deformed supersymmetry transformations

$$\begin{aligned}
\delta\lambda &= \sigma^{\mu\nu}\epsilon(F_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\overline{\lambda\lambda})\\
\delta A_\mu &= -i\overline{\lambda}\bar{\sigma}_\mu\epsilon\\
\delta F_{\mu\nu} &= i\epsilon(\sigma_\nu D_\mu - \sigma_\mu D_\nu)\overline{\lambda} + i(\nabla_\mu\epsilon\sigma_\nu - \nabla_\nu\epsilon\sigma_\mu)\overline{\lambda}\\
\delta\overline{\lambda} &= \frac{4\bar{\epsilon}}{\alpha'}\,,
\end{aligned} \tag{3.2}$$

where now, ϵ and $\bar{\epsilon}$ are the Killing spinors on euclidean $AdS_2 \times S^2$ satisfying the Killing equations,

$$\begin{aligned}
\nabla_\mu\epsilon^\alpha &= \frac{1}{\alpha'}C^{\alpha\beta}\sigma_{\mu\beta\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\\
\nabla_\mu\bar{\epsilon}_{\dot{\alpha}} &= \frac{1}{\alpha'}\,C_{\dot{\alpha}\dot{\beta}}\bar{\sigma}_\mu^{\dot{\beta}\alpha}\epsilon_\alpha\,.
\end{aligned} \tag{3.3}$$

Note that, for the last transformation in (3.2) to make sense, only the U(1) part of the $\overline{\lambda}$ field is meant to transform. So if we let $a = 1,2,3$ and $a = 4$ denote the SU(N) and U(1) gauge indices, respectively, then by the last transformation we mean $\delta\overline{\lambda}^a = 4\delta^{a4}\bar{\epsilon}/\alpha'$. Further note that, in the scaling limit of large k, if we choose $\epsilon \sim 1$ then the above equations imply that $\bar{\epsilon} \sim 1/k$. It is now straightforward to check that the action (3.1) is invariant under the transformations (3.2). Let us see this in some more details. The variation of terms present in the action are as follows:

$$\delta\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right) = -2i\epsilon F^{\mu\nu}\sigma_\nu D_\mu\overline{\lambda} - 2iF^{\mu\nu}\nabla_\mu\epsilon\,\sigma_\nu\overline{\lambda} \tag{3.4}$$

$$\delta\left(-2i\lambda\sigma^\mu D_\mu\overline{\lambda}\right) = -2i(\sigma^{\mu\nu}\epsilon)(F_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\overline{\lambda\lambda})\sigma^\rho D_\rho\overline{\lambda} - \frac{8i}{\alpha'}\lambda\sigma^\mu\nabla_\mu\bar{\epsilon} \tag{3.5}$$

$$\delta\left(-iC^{\mu\nu}F_{\mu\nu}\overline{\lambda\lambda}\right) = 2\epsilon\, C^{\mu\nu}\sigma_\nu D_\mu\overline{\lambda}(\overline{\lambda\lambda}) + 2C^{\mu\nu}\nabla_\mu\epsilon\sigma_\nu\overline{\lambda}(\overline{\lambda\lambda}) - \frac{8i}{\alpha'}C^{\mu\nu}F_{\mu\nu}(\bar{\epsilon}\overline{\lambda}) \tag{3.6}$$

$$\delta\left(\frac{1}{4}|C|^2(\overline{\lambda\lambda})^2\right) = \frac{4}{\alpha'}|C|^2(\overline{\lambda\lambda})(\bar{\epsilon}\overline{\lambda})\,. \tag{3.7}$$

Upon using the Killing spinor eqs. (3.3), we see that the two terms on the r.h.s. of (3.4), and the first term on the r.h.s. of (3.5) can be combined into

$$-2i\epsilon F^{\mu\nu}\sigma_\nu D_\mu\overline{\lambda} - 2iF^{\mu\nu}\nabla_\mu\epsilon\,\sigma_\nu\overline{\lambda} - 2iF_{\mu\nu}(\sigma^{\mu\nu}\epsilon)\sigma^\rho D_\rho\overline{\lambda} = \frac{8i}{\alpha'}C^{\mu\nu}F_{\mu\nu}(\bar{\epsilon}\overline{\lambda})\,, \tag{3.8}$$

which cancels the last term in eq. (3.6). The second term of (3.5) cancels the first term in (3.6)

$$C_{\mu\nu}\sigma^{\mu\nu}\epsilon\,\sigma^\rho D_\rho\overline{\lambda}(\overline{\lambda\lambda}) + 2\epsilon\, C^{\mu\nu}\sigma_\nu D_\mu\overline{\lambda}(\overline{\lambda\lambda}) = 0\,. \tag{3.9}$$

And finally, using (3.3) again, the second term of eq. (3.6) cancels the one in (3.7),

$$2C^{\mu\nu}\nabla_\mu\epsilon\,\sigma_\nu\overline{\lambda}(\overline{\lambda\lambda})+\frac{4}{\alpha'}|C|^2(\overline{\lambda\lambda})(\bar{\epsilon}\overline{\lambda})=0\,. \tag{3.10}$$

Also we note that the last term in (3.5) goes to zero in the limit $k\to\infty$. This completes the proof of invariance of the action under Q supersymmetry.

The modification we made in the transformation of $\overline{\lambda}$ is necessary for two reasons. Firstly, although it is of order $1/k$ and vanishes when $k\to\infty$, we have to keep it as it gives rise to some finite terms when acted on, for example, the third term in the action which is of order k. Secondly, this modification makes Q^2 vanish on shell. On the other hand, in the limit of $k\to\infty$, one gets back the usual $\mathcal{N}=1/2$ algebra.

4. Plane wave limit of $AdS_2\times S^2$

In this section, we study the plane wave limit of our supergravity setup. To consider this limit, we return to the Lorantzian version of $AdS_2\times S^2$ in the presence of graviphoton field, and take the limit for both metric and RR field. The $AdS_2\times S^2$ metric reads:

$$ds^2=R^2(-\cosh^2\!\rho\,d\tau^2+d\rho^2+\cos^2\theta\,d\psi^2+d\theta^2)\,. \tag{4.1}$$

We now switch to the light cone coordinates $\tilde{x}^\pm=(\tau\pm\psi)/2$, and do the following rescaling,

$$\begin{aligned} x^+ &= \tilde{x}^+\,, \qquad & r &= \rho R\,,\\ x^- &= \frac{\tilde{x}^-}{R^2}\,, \qquad & y &= \theta R\,. \end{aligned} \tag{4.2}$$

Taking the large limit of R, we arrive at the plane-wave limit of the metric and the RR-form field,

$$\begin{aligned} ds^2 &= -4dx^+dx^- -(r^2+y^2)(dx^+)^2+dr^2+dy^2\\ C &= dx^+\wedge dr+dx^+\wedge dy\,. \end{aligned} \tag{4.3}$$

In the vierbeins basis e^a_μ, the graviphoton field can be written as,

$$C^{\alpha\beta}=\frac{1}{2}C^{\mu\nu}(\sigma_{ab})^{\alpha\beta}e^a_\mu e^b_\nu\,. \tag{4.4}$$

Substituting the corresponding vierbeins, we find

$$\begin{aligned} C^{\alpha\beta} &= \frac{1}{2}(\sigma_{20}+\sigma_{30}+\sigma_{12}+\sigma_{13})^{\alpha\beta}\\ \bar{C}^{\dot\alpha\dot\beta} &= \frac{1}{2}(\sigma_{20}+\sigma_{30}+\sigma_{12}+\sigma_{13})^{\dot\alpha\dot\beta}\,, \end{aligned} \tag{4.5}$$

which, as shown in section 2, give rise to the non-(anti)commutativity relations between the odd coordinates

$$\begin{aligned} \{\theta^\alpha,\theta^\beta\} &= C^{\alpha\beta}\\ \{\bar\theta^{\dot\alpha},\bar\theta^{\dot\beta}\} &= \bar{C}^{\dot\alpha\dot\beta}\,. \end{aligned} \tag{4.6}$$

At first sight, it seems that the above anti-commutation relations break all the supersymmetries. However, note that in the plane wave limit (4.5) the determinant of $C^{\alpha\beta}$ (and $\bar{C}^{\dot{\alpha}\dot{\beta}}$) vanishes, which means that there exists a linear combination of θ^{α}'s for which some of the anticommutators are zero. Therefore, in principle, it should be possible to restore part of the supersymmetry algebra. In fact, it is possible to redefine the supercharges as

$$\begin{aligned}\hat{Q}_1 &= AQ_1 + BQ_2 \\ \hat{Q}_2 &= BQ_1 + AQ_2 \,,\end{aligned} \tag{4.7}$$

with some coefficients A and B, such that the algebra of supercharges is changed into a more attractive one,

$$\begin{aligned}\{\hat{Q}_2, \hat{Q}_2\} = \{\hat{Q}_1, \hat{Q}_2\} &= 0 \\ \{\hat{Q}_1, \hat{Q}_1\} &\neq 0 \,.\end{aligned} \tag{4.8}$$

The above algebra can be interpreted as the lorentzian $\mathcal{N} = 1/2$ SUSY algebra for which $\hat{Q}_1$ and $\hat{\bar{Q}}_{\dot{1}}$ are broken while $\hat{Q}_2$ and $\hat{\bar{Q}}_{\dot{2}}$ are survived. Unfortunately, since A and B in (4.7) depend on derivatives, the new charges are not linear in derivatives, and hence not obeying the Leibnitz's rule of derivation.

5. Summary and conclusion

In trying to extend the $\mathcal{N} = 1/2$ supersymmetric theory from flat space to a curved space as an effect of a graviphoton field, we observed that a graviphoton field with both self-dual and antiself-dual parts breaks the supersymmetry completely. On the other hand, taking a self-dual graviphoton field is not a solution to the supergravity equations in a curved background. However, we introduced a limit in which we could keep the background as euclidean $AdS_2 \times S^2$ while the antiself-dual part of the graviphoton is approaching zero. We showed in this limit, the C-deformed SYM theory, as introduced in [2] for flat space, can be lifted on the $AdS_2 \times S^2$ background. Further, by a small modification of the SUSY transformations, we proved the $\mathcal{N} = 1/2$ invariance of the theory.

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